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The moment map for circle actions on symplectic manifolds

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Dedicated to I.M. Gelfand
on the occasion of his 75th birthday

Abstract. *A circle action on a Kähler manifold has a moment map if and only if it has fixed points. In this note, we give examples to show that this statement is not true for a general symplectic manifold, though it does hold in dimension 4.*

§ 1. INTRODUCTION

Many problems in classical mechanics have a circular symmetry which can be used to reduce the number of degrees of freedom of the system by 2. This reduction is possible when the circle action is Hamiltonian, that is, when the inner product $i(\xi)\omega$ of the vector field ξ which generates the action with the symplectic form ω is exact. This means that there is a function μ (called the *moment* (or momentum) map) such that $i(\xi)\omega = d\mu$; and one can then reduce the phase space using μ by the procedure described in [MW]. This function μ has many other good properties. It is invariant under ξ; and is a perfect Morse function, which provides a nice decomposition of the symplectic manifold (W, ω) with one contribution from each component of the zero set of ξ. (See [F] for example. Note that zeros of ξ correspond to critical points of μ, and so always exist when

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W is compact). Various other consequences of the existence of a moment map are discussed by Ono in [O].

In light of this, it is of interest to find conditions which guarantee that a symplectic circle action be Hamiltonian. Clearly, it suffices that W be simply connected. Less obviously, it suffices that the circle action extend to the action of a semi-simple Lie group: see [MW]. In 1959, Frankel [F] discovered a beautiful theorem which applies in the Kähler case.

FRANKEL'S THEOREM. *A circle action which preserves the complex structure and the Kähler form on a compact Kähler manifold W is Hamiltonian if and only if it has fixed points.* ■

In this note we discuss the extent to which this result remains true for a circle action on a general compact symplectic manifold (W, ω). As we see from the following result, some extra hypotheses are needed to ensure that a circle action with fixed points be Hamiltonian.

PROPOSITION 1. *There is a 6-dimensional symplectic manifold (W, ω) with a symplectic circle action such that ξ has zeros but $i(\xi)\omega$ is not exact.*

The proof is given in §§2 and 3. It is based on an analysis of the structure of W when ξ does have zeros, using a generalization of the moment map. We also show:

PROPOSITION 2. *No such example can exist in dimension 4.* ■

There is one situation in which it is easy to generalize Frankel's theorem. Let us say that a 2n-dimensional symplectic manifold (W, ω) has «Lefschetz type» if $\wedge\omega^{n-1}$ induces an isomorphism from $H^1(W, \mathbb{R})$ to $H^{2n-1}(W, \mathbb{R})$. (Kähler manifolds are well known to satisfy this condition).

PROPOSITION 3. ([O]:Thm 4.1). *A symplectic circle action on a compact symplectic manifold of Lefschetz type is Hamiltonian if and only if it has fixed points.*

SKETCH OF PROOF. Consider the following conditions:

(i) $i(\xi)\omega$ is exact;

(ii) ξ has zeros;

(iii) the class $[\xi]$ in $H_1(W, \mathbb{R})$ which is represented by an orbit of ξ is zero.

Because W is compact and connected, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). It is not hard to check that the class $[\xi]$ in $H_1(W, \mathbb{R})$ is Poincaré dual to $[i(\xi)\omega^n]$. But the Lefschetz condition ensures that $[i(\xi)\omega^n] = 0$ if and only if $[i(\xi)\omega] = 0$.

Hence (iii) $\Rightarrow$ (i), and all three conditions are equivalent. ∎

As we see from Proposition 1, conditions (i) and (ii) above are not equivalent for a general symplectic manifold. Neither are conditions (ii) and (iii): in fact, we show below that any non-abelian symplectic nilmanifold has a free circle action satisfying (iii). (Many such manifolds are known: see [BG] and the references therein). Some other symplectic manifolds which are not nilmanifolds but which admit free circle actions satisfying (iii) are described in [Bou].

PROPOSITION 4. *Let W be a non-abelian compact nilmanifold $\Gamma\backslash G$ with a homogeneous symplectic form τ. (This means that τ lifts to a left-invariant form on G). Then (W, τ) admits a circle action which satisfies condition (iii) above but not (ii).*

Proof. If $\Gamma\backslash G$ is a compact nilmanifold, the center of Γ is contained in the center of G, so that each element of the center of Γ gives rise to a free circle action on $\Gamma\backslash G$: see [M]. Clearly, the homogeneous form τ is invariant under such an action. If G and Γ are not abelian, there is an element of the center of Γ which lies in the commutator subgroup $[\Gamma, \Gamma]$. This implies that the class $[\xi]$ represented by the orbits of the corresponding circle action is zero. ∎

We end this introduction by pointing out that Proposition 4 yields a quick proof of the following theorem on symplectic nilmanifolds, which was apparently first proved by Koszul (see [H]) and recently rediscovered by Benson-Gordon [BG].

PROPOSITION 5. *A compact symplectic nilmanifold of Lefschetz type is a torus.*

Proof. Proposition 3 and 4 together imply that if the symplectic form ω on $W = \Gamma\backslash G$ is homogeneous then G is abelian. Hence W is a torus. As remarked in [BG], this argument may be extended to non-homogeneous forms by the use of Nomizu's theorem, which says that any cohomology class $[\omega]$ on $\Gamma\backslash G$ may be represented by a homogeneous form τ. For, if the homogeneous form τ is cohomologous to the symplectic form ω, the class $[\tau^n] = [\omega^n]$ is non-zero. Because a top-dimensional homogeneous form must either vanish everywhere or nowhere, this means that τ^n never vanishes. Hence W has a homogeneous symplectic form τ and we can apply the previous argument to (W, τ). ∎

§2. THE GENERALIZED MOMENT MAP

In this section we prove Proposition 2 and begin the proof of Proposition 1.

The main idea is to analyse what happens when one passes through a critical level of the (generalized) moment map. We shall only give a detailed treatment of the case when the zero sets Z of ξ have codimension 4, since this is all we need here. Guillemin and Sternberg [GS] have complete results in the general case.

We begin by defining the generalized moment map. If the class $[i(\xi)\omega]$ is non-zero and integral, there is by [T] a map $\psi : W \to S^1$ such that $i(\xi)\omega = \psi^*(ds)$. We call ψ a *generalized moment map* for ω. As we will see below, it has many of the properties of an ordinary moment map (and may even be used to reduce W). First, we show:

LEMMA 1. *Let ω be an S^1-invariant symplectic form on W such that $[i(\xi)\omega]$ is non-zero. Then W carries an S^1-invariant symplectic form which admits a generalised moment map ψ.*

Proof. Observe that the class $[i(\xi)\omega]$ is rational if $[\omega]$ is. For the value of $[i(\xi)\omega]$ on a loop λ equals the value of $[\omega]$ on the 2-cycle $[\phi_t(\lambda)\colon 0 \leqslant t \leqslant 1]$, where ϕ_t is the flow of ξ. Now, if $[\omega]$ is not rational, there is always a symplectic form whose cohomology class is rational and which is so close to ω that its average $\hat{\omega}$ over S^1 is symplectic and satisfies $[i(\xi)\hat{\omega}] \neq 0$. Thus a multiple of $\hat{\omega}$ admits a generalized moment map. ■

Clearly, the critical points of ψ are exactly the zeros of ξ.

LEMMA 2. *The set of critical points of ψ is a disjoint union of symplectic submanifolds of W each of codimension at least 4.*

Proof. First observe that there is always an S^1-invariant Riemannian metric g on W which is compatible with ω in the sense that $g(.\ ,.) = \omega(.\ , J.)$, where J is an S^1-invariant almost complex structure on W. (Such a metric is sometimes called an almost Kähler metric). This follows because any S^1-invariant metric $\hat{g}$ is related to ω by the identity $\hat{g}(.\ ,.) = \omega(.\ , A\ .)$, for a unique A which is non-singular, skew-symmetric and S^1-invariant. Then $-A^2 = \Lambda$ is positive definite and S^1-invariant, and so $g(.\ ,.) = \hat{g}(.\ , \Lambda^{-1/2}A\ .)$ has the desired properties.

If we now identify S^1 with $\mathbb{R}/\mathbb{Z}$ in the usual way, we may define the gradient vector field of ψ with respect to g. It is easy to check that this is just $J\xi$, so that it commutes with ξ. (Observe that $[\xi, J\xi] = \mathcal{L}_\xi(J\xi) = \mathcal{L}_\xi(J)\xi + J\mathcal{L}_\xi\xi = 0$). This flow has all the nice properties possessed by the gradient flow of an ordinary moment map.: see [F]. In particular, its critical set is a disjoint union of symplectic submanifolds Z. The normal bundle of Z has a complex structure induced by

J and splits as a sum $\nu^- \oplus \nu^+$, where ν^- is tangent to the incoming flow lines of $J\xi$ (i.e. the stable manifold) and is the subbundle where the S^1-action rotates in the anticlockwise direction. Thus the indices of the critical submanifolds are all even: in particular, there are no critical submanifolds of index or co-index 1. As noted by Atiyah in [A], this implies that the number of components of the level sets of ψ changes only when one passes a local maximum or minimum. Because the map ψ is essential, it follows easily that the number of components is constant. Thus ψ has no local maxima or minima, and the codimension of each Z is least 4. ∎

Let us now specialize to the case when dim $W = 4$. Then each critical point is isolated, and has a neighbourhood of the form $D^2 \times D^2$ with S^1-action given by $(z, w) \to (e^{2\pi i p\theta} z, e^{-2\pi i q\theta} w)$, where D^2 is a little disc with center 0 in $\mathbb{C}$, and p and q are greater than 0. Such a point will be said to have type (p, q). The non-critical level sets F of ψ are S^1-invariant, and the quotient map $\pi : F \to F/S^1$ is a Seifert fibration whose base $B = F/S^1$ is an orbifold (or V-manifold). This means that B is a topological 2-manifold which has a differentiable structure with a finite number of conical singularities. As one passes a critical level in the direction of $J\xi$, F changes by a finite number of surgeries, each given by attaching the above 3-sphere $\partial(D^2 \times D^2)$ equivariantly to F along $(\partial D^2) \times D^2$. It is possible to assign to each Seifert fibration $\pi : F \to B$ a rational number $\chi(F)$ called the Euler number which generalizes the usual Euler number of a circle bundle over a 2-manifold. We claim:

LEMMA 3. *Let F_2 be obtained from F_1 by one surgery of type (p, q). Then* $\chi(F_2) = \chi(F_1) - 1/pq$.

Proof. According to [Th], one can calculate $\chi(F)$ as follows. Choose an S^1-invariant 1-form α such that $\alpha(\xi) = 1$ everywhere. Then $\chi(F)$ is just the integral of $-\alpha \wedge d\alpha$ over F. One must be careful about orientations here. We will orient F so that the vectors ξ, v_1, v_2 form a positively oriented basis on F if v_1 and v_2 project to a positively oriented basis on B. Then, the above definition of $\chi(F)$ agrees with the usual one when $F \to B$ is a fibration, since in this case $-d\alpha$ pushes down to a 2-form on B which represents the first Chern class.

Now, α can be constructed as a sum $\Sigma(\rho_i \circ \pi)\beta_i$, where $[\rho_i]$ is a partition of unity of B subordinate to some covering U_i and the β_i are suitable 1-forms on the sets $\pi^{-1}(U_i)$. In particular, one can choose α_1 on F_1 so that it equals $d\theta/p$ in the image of $(\partial D^2) \times D^2$, where we use the polar coordinates (r, θ) and (ρ, ϕ) in the two copies of D^2, normalised so that $0 \leqslant \theta, \phi \leqslant 1$. Thus $(\partial D^2) \times D^2$ does not contribute to $\chi(F_1)$. Next observe that F_2 equals F_1 with $(\partial D^2) \times$

$\times D^2$ replaced by $D^2 \times (\partial D^2)$. Let α_2 equal α_1 outside $D^2 \times (\partial D^2)$, and equal β inside $D^2 \times (\partial D^2)$ where

$$\beta = \lambda(\gamma)\, d\theta/p - (1 - \lambda(\gamma))\, d\phi/q,$$

and where λ is a smooth function which is 0 near $r = 0$ and 1 near $r = 1$. Then $\beta(\xi) = 1$ and the integral of $\beta \wedge d\beta$ over $D^2 \times (\partial D^2)$ is $1/pq$. (Note that the D^2 factor in $D^2 \times (\partial D^2)$ has the usual orientation, since the restriction of ω to the discs $D^2 \times pt \subset F_2$ is symplectic. However, ∂D^2 is oriented according to the action of S^1, so that the integral of $d\phi$ over pt $\times (\partial D^2)$ is -1). ∎

Proof of Proposition 2.

Suppose that (W, ω) is a 4-dimensional symplectic manifold with an S^1-action such that $i(\xi)\omega$ is not exact. By Lemma 1 we may assume that ω has a generalized moment map ψ. Consider the regular level sets $F_s = \psi^{-1}(s)$ as s moves around S^1. Clearly, $\chi(F_s)$ is constant except when s passes through a critical point in which case, by Lemma 3, it decreases. Since after going round the whole circle one must eventually return to the start, there cannot be any critical points at all. ∎

Let us now go back to the general case and look at the structure near the regular points of ψ. For simplicity, we will assume that there are no finite isotropy groups. Let $\mathscr{I} \subset S^1$ be a connected arc consisting of regular values of ψ. Then, for each $s \in \mathscr{I}$, the level set $F_s = \psi^{-1}(s)$ is diffeomorphic to the total space $F_{\mathscr{I}}$ of a circle bundle $\pi : F_{\mathscr{I}} \to B$, whose first Chern class we will call $c_{\mathscr{I}}$. Moreover, $\psi^{-1}(\mathscr{I})$ is S^1-equivariantly diffeomorphic to $F_{\mathscr{I}} \times \mathscr{I}$, and any S^1-invariant symplectic form ω on $\psi^{-1}(\mathscr{I})$ with moment map ψ may be written as:

$$\omega = \pi^*(\tau_s) + \beta_s \wedge ds \tag{a}$$

where τ_s is a family of symplectic forms on B and β_s is a family of S^1-invariant 1-forms on $F_{\mathscr{I}}$ with $\beta_s(\xi) = 1$. As in Lemma 3, each $d\beta_s$ is pulled back from a form γ_s on B which represents the cohomology class $-c_{\mathscr{I}}$. Further, because ω is closed, $d/ds\, (\pi^*\tau_s) = -d\beta_s = -\gamma_s$. Therefore, for s and t in $\mathscr{I}$, we have

$$[\tau_t] = [\tau_s] + (t - s)\, c_{\mathscr{I}}. \tag{b}$$

(See [DH]). Conversely, given any family of symplectic forms τ_s on the base B which satisfy (b), one can, by [K], find 1-forms β_s on $F_{\mathscr{I}}$ for which $d\beta_s = -\dfrac{d}{ds}(\pi^*\tau_s)$. Then formula (a) defines an S^1-invariant symplectic form ω on $\psi^{-1}(\mathscr{I})$ with moment map ψ.

LEMMA 4. *ω is determined by the forms τ_s up to an S^1-equivariant diffeomorphism which preserves the level sets of ψ.*

Proof. Different choices of the β_s differ at most by a family of forms $\pi^*(\eta_s)$, where each η_s is closed on B. Therefore the forms

$$\omega_t = \pi^*(\tau_s) + \beta_s \wedge ds + t\pi^*(\eta_s) \wedge ds$$

are symplectic for $0 \leqslant t \leqslant 1$, and it sufficies to construct a family g_t of S^1-equivariant diffeomorphisms of $\psi^{-1}(\mathscr{I})$ which preserve the fibers F_s and are such that $g_t{}^*(\omega_t) = \omega_0$. Now,

$$\frac{d}{dt}(\omega_t) = \pi^*(\eta_s) \wedge ds = d[\pi^*(\lambda_s)] \text{ where } \lambda_s = \int^s \eta_r \, dr.$$

Let ζ_s be the vector field on B such that $i(\zeta_s)\tau_s + \lambda_s = 0$, and for each t let ζ_s^t be the unique lift of ζ_s to F_s which lies in the kernel of the 1-form $\beta_s + t\pi^*(\eta_s)$. Then, for each t, the ζ_s^t fit together to form an S^1-invariant vector field ζ^t on $\psi^{-1}(\mathscr{I})$ such that

$$d[i(\zeta^t)\,\omega_t] + \frac{d}{dt}(\omega_t) = 0.$$

It follows that the flow g_t of ζ^t has all the desired properties. ∎

Finally, let us look at what happens in a neighbourhood P of a critical submanifold Z of codimension 4. We will suppose that $P = \psi^{-1}([\lambda - \epsilon, \lambda + \epsilon])$, where λ is a critical value, and write $F_s = \psi^{-1}(s)$ as before.

LEMMA 5 (i). *If Z has codimension 4 and there are no finite isotropy groups, then the base manifolds F_s/S^1, $s \neq \lambda$, are all diffeomorphic, to B say. Moreover the projections $F_s \to B$ fit together to give a smooth map $\hat{\pi} : P \to B$.*
(ii) *Let c_s denote the Chern class of $F_s \to B$. Then $c_{\lambda+\epsilon} = c_{\lambda-\epsilon} - D(Z)$, where $D(Z)$ is the Poincaré dual of Z in B.*

Proof. (i). It suffices to prove the second statement. We will use the notation of Lemma 2. Observe first that a neighbourhood U of the zero section in the normal bundle $\nu = \nu^- \oplus \nu^+$ of Z in P has a symplectic form τ, which is invariant under the structure group $S^1 \times S^1$ and which restricts on each fiber to $i/2[du \wedge d\bar{u} + dv \wedge d\bar{v}]$, where (u, v) are coordinates on the fibers $\mathbb{C} \oplus \mathbb{C}$. The equivariant symplectic neighbourhood theorem then implies that a neighbourhood $N(Z)$ of Z in P is equivariantly symplectomorphic to (U, τ) with its obvious circle action. If we now choose J on $N(Z)$ so that it restricts on each fiber to multiplication by i, the vector field $J\xi$ is just $-\rho_u \oplus \rho_v$, where ρ_u and ρ_v are the radial fields $Re(u\partial_u)$ and $Re(v\partial_v)$.

Now, let $S(Z) \subset P$ be the set of points which flow into Z under $J\xi$, so that

$S(Z) \cap N(Z) \subset \nu^- \oplus 0$. Then we can define a smooth map $\tilde{\pi}$ from $P - S(Z)$ to B by flowing a point x along $J\xi$ until it reaches F_ϵ and then projecting it to B. This extends continuously to P since each S^1-orbit in $S(Z)$ corresponds to a unique point of Z and hence to a unique S^1-orbit in F_ϵ. It obviously suffices to check smoothness near a point $z \in Z$. But, the normal bundle of $\tilde{\pi}(Z)$ in B is clearly $\nu^- \otimes \nu^+$ and one can check, using the explicit formula for $J\xi$ above, that the map $\tilde{\pi}$ on $N(Z)$ is given by $(z, u, v) \to (z, uv)$, which is smooth.
(ii) When $\dim W = 4$, this was proved in Lemma 3. The proof of the general case is similar and will be left to the reader. ■

This lemma gives us some idea of what the example of Proposition 1 must look like. For suppose that (W, ω) is a symplectic manifold with an S^1-action which has no finite isotropy groups and whose critical manifolds Z_i all have codimension 4. If the action is not Hamiltonian, we may assume by Lemma 1 that there is a generalized moment map $\psi : W \to S^1$. If in addition one of the Chern classes $c_{\mathscr{g}}$ vanishes, then Lemma 5 implies that the sum of the $D(Z_i)$'s must vanish. In particular, the Z_i cannot be τ_s-symplectic for all s, even though each Z_i is τ_s-symplectic for s near the corresponding critical level λ_i. Thus, the τ_s must vary significantly as s goes round the circle. Note that this cannot happen when W is Kähler and ξ is holomorphic, since in that case $J\xi$ is holomorphic too, and the reduced space B has an induced complex structure with respect to which the forms τ_s are Kähler and the manifolds Z_i holomorphic.

§3. THE 6-DIMENSIONAL EXAMPLE

We construct a symplectic manifold (X, ω), which admits a circle action with moment map $\mu : X \to [0, 7]$. The manifold X has two boundary components which lie over the endpoints 0 and 7, and we form W by glueing them together. Further, X has four critical levels at $s = 1, 2, 5$ and 6 with zero sets of codimension 4, and no finite isotropy groups. Therefore, by Lemmas 4 and 5, X projects smoothly onto the base manifold B, and to describe ω we need only define a suitable family of symplectic forms τ_s on B for regular values of s, and then describe what heppens near the singular levels.

Let $B = T^4$ with coordinates x^1, x^2, x^3, and x^4, and denote the form $dx^i \wedge \wedge\, dx^j$ by σ_{ij}. Then the forms τ_s and Chern classes $c_{\mathscr{g}}$ are listed below, where $K > 2$ will be chosen later. We also list relevant data at the singular fibers, using the notation L_{ij} to denote a 2-torus on which the two coordinates other than x^i and x^j are constant.

for $s \in [0, 1)$,	$\tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42}$,	$c_{\mathscr{I}} = 0$
at $s = 1$,	$Z = L_{13}$,	$D(Z) = [\sigma_{42}]$
for $s \in (1, 2)$,	$\tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + (3 - s)\sigma_{42}$,	$c_{\mathscr{I}} = -[\sigma_{42}]$
at $s = 2$,	$Z = L_{42}$,	$D(Z) = [\sigma_{13}]$
for $s \in (2, 5)$,	$\tau_s = K\sigma_{12} + K\sigma_{34} + (4 - s)\sigma_{13} + (3 - s)\sigma_{42}$,	$c_{\mathscr{I}} = -[\sigma_{31} + \sigma_{42}]$
at $s = 5$,	$Z = L_{31}$,	$D(Z) = -[\sigma_{42}]$
for $s \in (5, 6)$,	$\tau_s = K\sigma_{12} + K\sigma_{34} + (4 - s)\sigma_{13} - 2\sigma_{42}$,	$c_{\mathscr{I}} = -[\sigma_{31}]$
at $s = 6$,	$Z = L_{24}$,	$D(Z) = -[\sigma_{13}]$
for $s \in (6, 7]$,	$\tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - 2\sigma_{42}$.	$c_{\mathscr{I}} = 0$.

It is easy to check that this information is all compatible. Note also that, for $\mathscr{I} = [0, 1)$ and $(6, 7]$, $\mu^{-1}(\mathscr{I})$ is the product $T^4 \times S^1 \times \mathscr{I}$ with form ω equal either to $(K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42}) \oplus d\theta \wedge ds$ or to $(K\sigma_{12} + K\sigma_{34} + 2\sigma_{31} + 2\sigma_{24}) \oplus d\theta \wedge ds$. Hence $\mu^{-1}(0)$ may be glued to $\mu^{-1}(7)$ by the diffeomorphism of T^4 which interchanges x^1 with x^3 and x^2 with x^4.

We next describe what happens near the critical levels. For $\lambda = 1, 2, 5$ and 6 we will construct a piece P_λ of symplectic manifold which lies over $[\lambda - \epsilon, \lambda + \epsilon]$ and glues to the parts of X which are already defined. By Lemma 4, P_λ will glue to X provided that the forms τ_s on the base agree. In fact, it suffices to do this at $\lambda = 1$ and 2 since the diffeomorphism of T^4 which interchanges x^1 with x^4 and x^2 with x^3 takes τ_s to $-\tau_{7-s}$. Thus the singularity as s increases through 1 is diffeomorphic to the singularity as s decreases through 6, and similarly for 2 and 5.

The singularity at $s = 1$

We define P_1 as a product $L_{13} \times Y$ where Y is a 4-dimensional symplectic manifold with an S^1 action which has a point singularity. To construct Y, let us first consider the manifold S^2 with a symplectic form ρ of total area 1 which is invariant under the usual action of S^1 by rotation. Then the moment map takes S^2 onto $[0, 1]$ and has one minimum at m say, and one maximum at M. Next consider $S^2 \times S^2$ with symplectic form $2\rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are copies of ρ, and with the diagonal action of S^1. Then the moment map μ is $2\mu_1 + \mu_2$ where μ_i is the moment map for the ith factor with respect to ρ_i. There are now four fixed points which occur at 0, 1, 2, and 3 in the order $m \times m$, $m \times M$, $M \times m$ and $M \times M$. It is easy to check that the level sets for s between

$1 = \mu(m \times M)$ and $2 = \mu(M \times m)$ are diffeomorphic to $S^1 \times S^2$ since, for each point on the second sphere there is a circle of possible points on the first sphere. Thus the corresponding circle bundle is a product. In contrast, the level sets for $s > 2$ are 3-spheres, and, by Lemma 3, the circle bundle has Euler characteristic -1. Set $V = \mu^{-1}([2 - \epsilon, 2 + \epsilon]) \subset S^2 \times S^2$ with the induced symplectic form, where $0 < \epsilon < 1$. By Lemma 5, V projects onto S^2. Cut out from V the inverse image of a 2-disc in S^2 which avoids the unique critical value of this projection. This inverse image is an S^1-invariant set diffeomorphic to $D^2 \times S^1 \times \times [-\epsilon, \epsilon]$ with product symplectic form, and we may glue back in its place a copy of $(T^2 - \mathrm{Int}(D^2)) \times S^1 \times [-\epsilon, \epsilon]$ with product symplectic form $\sigma \oplus d\theta \wedge ds$. This last is the manifold we call Y. Note that $Y \cap \mu^{-1}([-\epsilon, 0))$ is a product $T^2 \times S^1 \times [-\epsilon, 0)$. Clearly, we may choose σ so that the induced symplectic form σ on Y integrates to 1 over $T^2 \times pt. \times [-\epsilon]$. It follows from formula (b) above that, when $s > 0$, the induced form τ_s on the base T^2 has integral $1 - s$.

By Lemma 5, there is a smooth map $\tilde{\pi}$ of Y onto $T^2 \times [-\epsilon, \epsilon]$ whose fibers are the orbits of S^1. Thus, if x^4 and x^2 are coordinates on T^2 the forms dx^4 and dx^2 pull back to 1-forms which we will call α^4 and α^2 on Y. We now set P_1 equal to the product $T^2 \times Y$ with symplectic form $\omega_1 = Kdx^1 \wedge \alpha^2 + + Kdx^3 \wedge \alpha^4 + 2dx^1 \wedge dx^3 + 2\tilde{\sigma}$, where x^1 and x^3 are the coordinates on the first T^2-factor. It is easy to check that ω_1 is indeed symplectic. Moreover P_1 does glue to X. As the notation indicates, the T^2 factor in P_1 should be identified with L_{13} in T^4 and the base T^2 of Y with L_{42}.

The singularity at $s = 2$

This time ν^- is not trivial, for the circle bundle $S(\nu^-) \to Z = L_{42}$ may be identified with the restriction of the bundle $F_s \to T^4$ to L_{42}, and so it has Euler characteristic -1. We put $P_2 = S(\nu^-) \times_{S^1} Y$, so that P_2 fibers over L_{42} with fiber Y. The quotient space of P_2 by the S^1-action is the manifold $L_{42} \times T^2 \times \times [-\epsilon, \epsilon]$, and it is easy to check that the Chern class of the S^1-bundle at $-\epsilon$ is $-[\sigma_{42}]$ while that at ϵ is $-[\sigma_{42} + \sigma_{13}]$, where we identify the T^2 factor with L_{13}. We will denote the lift of the forms dx^i to P_2 by α^i, for $i = 1, \ldots, 4$.

Our next task is to extend the form $\tilde{\sigma}$ over P_2. Note that $\tilde{\sigma} - \alpha^1 \wedge \alpha^3$ is exact, so that it may be written as $d\gamma$ for some 1-form γ which we may choose to vanish at the unique singular point of Y. Consider the 1-form

$$\rho = \Sigma_i(\lambda_i \circ pr)\, \gamma_i,$$

on P_2, where pr is the fibration $P_2 \to L_{42}$, $[\lambda_i]$ is a partition of unity subordinate to a trivializing cover $[U_i]$ for this fibration, and γ_i is the pull-back to $U_i \times Y \cong \cong pr^{-1}(U_i) \subset P_2$ of γ. Since the local trivializations must fixed the unique critical point in each fiber, $i(v)\, d\rho = 0$ for any vector v tangent to Z. Also, $d\rho$ restricts to $\tilde{\sigma} - \alpha^1 \wedge \alpha^3$ on each fiber Y.

Now consider the form

$$\omega_2 = K\alpha^1 \wedge \alpha^2 + K\alpha^3 \wedge \alpha^4 + 2\alpha^1 \wedge \alpha^3 + \alpha^4 \wedge \alpha^2 + 2d\rho$$

on P_2. We claim that, if K is sufficiently large, this is symplectic. For it is clearly non-degenerate on the restriction of TP_2 to Z, and hence near Z. And, away from Z, one can use the argument of [McD 2] Lemma 3.3. Further, one can check directly that the reduced forms $\hat{\tau}_s$ on the base $L_{42} \times L_{13}$ are cohomologous to τ_{2+s} for $-\epsilon < s < 0$. It follows from Lemma 5 that this must also hold for $0 < s < \epsilon$. Since $\hat{\tau}_s - \tau_{2+s}$ depends only on $d\rho$, one can choose a large K so that the forms $t\,\hat{\tau}_s + (1-t)\,\tau_{2+s}$ are symplectic for t between 0 and 1 and s between $-\epsilon$ and $-\epsilon/2$ and $\epsilon/2$ and ϵ. Moser's theorem then implies that the forms $\hat{\tau}_s$ and τ_{2+s} are diffeomorphic for each s in the given range, so that P_2 can be patched to X as required.

This completes the construction, and hence the proof of Proposition 1.

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